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LETTER TO THE EDITOR

Finite-size corrections in the non-linear Schrödinger model

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Abstract. A method based on the Euler-Maclaurin formula is proposed to obtain all finite-size corrections to the energies of the ground and excited states for the one-dimensional Bose gas with delta function interactions. Scaling dimensions for all gapless states and some states having a macroscopic momentum are obtained.

The one-dimensional Bose gas with pairwise repulsive delta function interactions (the so-called non-linear Schrödinger (NLS) model) is known to be an exactly integrable dynamical system with a second quantised Hamiltonian

$$\hat{H} = \int_0^L \mathrm{d}x \left(\frac{\partial \phi^\dagger}{\partial x} \frac{\partial \phi}{\partial x} + g : \phi^\dagger \phi \phi^\dagger \phi : - \zeta : \phi^\dagger \phi : \right) \tag{1}$$

where $g>0$ is the strength of the contact interaction and ζ is the chemical potential. The field operators satisfy canonical commutation relations

$$\begin{aligned} &[\phi(x), \phi(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0 \\ &[\phi(x), \phi^\dagger(y)] = \delta(x-y). \end{aligned} \tag{2}$$

This model has been solved by the Bethe ansatz technique [1] and more recently with the help of the quantum inverse scattering method [2]. Explicit Green function calculations [3, 4] show that the NLS model undergoes a phase transition at zero temperature. Therefore, one expects this model to exhibit local scale (conformal) invariance at zero temperature for distances much larger than the inverse Fermi momentum.

The Hilbert space of a field theory possessing conformal invariance has two sets of mutually commuting Virasoro operators, which are the generators of conformal transformations

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}c\, m(m^2-1)\delta_{m+n,0} \\ &[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \tfrac{1}{12}c\, m(m^2-1)\delta_{m+n,0} \end{aligned} \tag{3}$$

where c is called the conformal anomaly. The L_n and $\bar{L}_n$ are the coefficients in a Laurent expansion of the analytic and antianalytic pieces of the stress tensor

$$T(z) = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}. \tag{4}$$

Unlike the situation for $c<1$ [5-7], for $c \geq 1$, unitarity does not constrain either c or the scaling dimensions of the primary operators (building blocks from which other operators can be constructed). In fact, for $c=1$ one expects to find an infinite set of primary operators with non-trivial dimensions which depend on the coupling constant and at least one operator with dimension two (the marginal operator) which moves the system along the line of fixed points.

In parallel with the formal developments [5, 6], Cardy and collaborators [8] showed that finite-size corrections to the eigenvalues of the transfer matrix (or the quantum Hamiltonian) of a conformally invariant model were directly related to the conformal anomaly and the scaling dimensions of operators in the model. More precisely

$$\frac{E_0(L)}{L}=e_\infty-\frac{\pi c N_f}{6L^2}+\mathrm{O}\left(\frac{1}{L^3}\right) \tag{5}$$

where $E_0(L)$ is the ground-state energy of a system of size L, e_∞ is the energy per unit length in the thermodynamic limit, c is the conformal anomaly and N_f is a normalisation factor. N_f is defined in such a way that the equations of motion are conformally invariant.

Also for each operator O_α with scaling dimension x_α and spin s_α, there exists a tower of states in the spectrum of $\hat{H}$ with energies $E^\alpha_{jj'}(L)$ and momenta $P^\alpha_{jj'}(L)$ given by

$$\begin{aligned} E^\alpha_{jj'}(L)&=E_0(L)+\frac{2\pi(x_\alpha+j+j')}{L}N_f+\mathrm{O}\left(\frac{1}{L^2}\right)\\ P^\alpha_{jj'}(L)&=\frac{2\pi(s_\alpha+j-j')}{L}+\mathrm{O}\left(\frac{1}{L^2}\right)\end{aligned} \tag{6}$$

where j, j' are non-negative integers.

For exactly solvable conformally invariant models these equations have recently been used to calculate exact scaling dimensions of various operators [9–14]. Especially interesting is the method of Woynarovich and Eckle [14] based on the Euler–Maclaurin formula. This method can be applied to find all higher-order finite-size corrections in a systematic way.

In this letter we apply the Euler–Maclaurin formula

$$\sum_1^n f(n)=\int_1^n f(x)\mathrm{d}x+\tfrac{1}{2}(f(1)+f(n))+\sum_1^\infty\frac{B_{2p}}{2p!}(f^{(2p-1)}(n)-f^{(2p-1)}(1)) \tag{7}$$

to analyse the finite-size corrections to the Bethe ansatz equations for the NLS model, which, for periodic boundary conditions, are [1]

$$\mathrm{e}^{\mathrm{i}\lambda_j L}=\prod_{k\neq j}\left(\frac{\lambda_j-\lambda_k+\mathrm{i}g}{\lambda_j-\lambda_k-\mathrm{i}g}\right) \tag{8}$$

where the pseudomomenta λ_j are all real and distinct. Taking the logarithm (with an appropriate choice of branch) we get a system of coupled equations that the pseudomomenta satisfy:

$$\lambda_j L+\sum_{k=1}^N\theta(\lambda_j-\lambda_k)=2\pi n_j$$

where

$$\theta(\lambda)=\begin{cases}\pi-2\tan^{-1}g/\lambda & \lambda>0\\ -\pi+2\tan^{-1}g/(-\lambda) & \lambda<0\end{cases} \tag{9}$$

and n_j are a set of distinct integers (for odd numbers of particles N) or half-odd integers (for even N). The energy and momentum are

$$E=\sum_{j=1}^{N}(\lambda_j^2-\zeta)$$
$$P=\sum_{j=1}^{N}\lambda_j. \tag{10}$$

For the ground state the n_j are consecutive integers (or half-odd integers) lying in the range $-\frac{1}{2}(N-1)\leq nj\leq\frac{1}{2}(N-1)$. We define, after de Vega and Woynarovich [11], a function $z_N(L,\lambda)$

$$z_N(L,\lambda)=\frac{1}{2\pi}\left(\lambda+\frac{1}{L}\sum_k\theta(\lambda-\lambda_k)\right) \tag{11}$$

where the λ_k are the solutions to (roots of) the Bethe ansatz questions. $Lz\ (L,\lambda)$ becomes an integer (or half-odd integer) when λ coincides with a root.

We can now define a density for the roots as

$$\rho_N(L,N)=\frac{\mathrm{d}z_N(L,\lambda)}{\mathrm{d}\lambda}. \tag{12}$$

In the thermodynamic limit ($L, N\to\infty$; $N/L=D=$ density kept fixed) the root density $\rho_\infty(\lambda)$ satisfies a linear integral equation [1]

$$\rho_\infty(\lambda)-\frac{1}{2\pi}\int_{-q}^{q}K(\lambda,\mu)\rho_\infty(\mu)\,\mathrm{d}\mu=\frac{1}{2\pi}$$

where

$$K(\lambda,\mu)=2g/[g^2+(\lambda-\mu)^2] \tag{13}$$

and q is the Fermi momentum which is fixed by the condition that the configuration of consecutive integers mentioned above be a true minimum of the energy.

To analyse the finite case, we apply the Euler–Maclaurin formula to (9) to get, for the ground state, to order $1/L^2$

$$\begin{aligned}\frac{\lambda(L,z)}{2\pi}&+\frac{1}{2\pi}\int_{-z_m+1/2L}^{z_m-1/2L}\theta[\lambda(L,z)-\lambda(L,z')]\,\mathrm{d}z'\\&+\frac{1}{4\pi L}\{\theta[\lambda(L,z)-\lambda(L,z_m-1/2L)]+\theta[\lambda(L,z)-\lambda(L,-z_m+1/2L)]\}\\&-\frac{1}{24\pi L^2}\{K[\lambda(L,z)-\lambda(L,z_m-1/2L)]\lambda'(L,z_m-1/2L)\\&-K[\lambda(L,z)-\lambda(L,-z_m+1/2L)]\lambda'[L,-z_m+1/2L)\}+\mathrm{O}\left(\frac{1}{L^3}\right)=z\end{aligned} \tag{14}$$

where $z_m=N/2L=D/2$.

We can now extend the limits of integration to $[-z_m,z_m]$ and remove the extra integrals:

$$\frac{1}{2\pi}\int_{-z_m+1/2L}^{z_m-1/2L}\theta\,\mathrm{d}z'=\frac{1}{2\pi}\int_{-z_m}^{z_m}\theta\,\mathrm{d}z'-\frac{1}{2\pi}\int_{z_m-1/2L}^{z_m}\theta\,\mathrm{d}z'-\frac{1}{2\pi}\int_{-z_m}^{-z_m+1/2L}\theta\,\mathrm{d}z'. \tag{15}$$

$z(L, \lambda)$ is an analytic function since it is the sum of finitely many analytic functions. Thus its inverse $\lambda(L, z)$ is also analytic and we can expand the integrand in the two extra integrals in a Taylor series around $z' = \pm z_m$. (In this model even $z_\infty(\lambda)$ and $\lambda_\infty(z)$ are analytic [1].) We finally get

$$\frac{\lambda(L, z)}{2\pi} + \frac{1}{2\pi} \int_{-z_m}^{z_m} \theta(\lambda(L, z) - \lambda(L, z'))\, \mathrm{d}z' = z - \frac{1}{48\pi L^2} \frac{K(\lambda, q) - K(\lambda, -q)}{\rho_\infty(q)}. \tag{16}$$

We now make the ansatz

$$\lambda(L, z) = \lambda_\infty(z) + \frac{g_2(z)}{L^2} + \mathrm{O}\left(\frac{1}{L^3}\right) \tag{17}$$

which, after changing variables from z to $\lambda_\infty(z)$ and subtracting a similar equation for the infinite case, leads to an integral equation for g_2:

$$(\rho_\infty g_2)(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu)\, (\rho_\infty g_2)(\mu)\, \mathrm{d}\mu = -\frac{1}{48\pi} \frac{K(\lambda, q) - K(\lambda, -q)}{\rho_\infty(q)}. \tag{18}$$

The integral operator on the left can be written in the form $(1 - \hat{K}/2\pi)$ where

$$\hat{K}(f)(\lambda) = \int_{-q}^{q} K(\lambda, \mu) f(\mu)\, \mathrm{d}\mu. \tag{19}$$

We now introduce the inverse operator in the form $(1 + \hat{M})$ where

$$(\hat{M}f)(\lambda) = \int_{-q}^{q} M(\lambda, \mu) f(\mu)\, \mathrm{d}\mu. \tag{20}$$

The kernel M satisfies the equation

$$M(\lambda, \mu) - \frac{1}{2\pi} \int_{-q}^{q} M(\lambda, \nu) K(\nu, \mu)\, \mathrm{d}\nu = \frac{K(\lambda, \mu)}{2\pi}. \tag{21}$$

Using the properties of the kernel we get

$$(\rho_\infty g_2)(\lambda) = -\frac{1}{24} \frac{M(\lambda, q) - M(\lambda, -q)}{\rho_\infty(q)}. \tag{22}$$

In terms of this function the finite-size root density can be written as

$$\rho(L, \lambda) = \rho_\infty(\lambda) - \frac{(\rho_\infty g_2)'(\lambda)}{L^2} + \mathrm{O}\left(\frac{1}{L^3}\right). \tag{23}$$

To compute the finite-size correction to the ground-state energy we apply the Euler–Maclaurin formula to the energy (equation (10)) and after the usual manipulations we get

$$E_0(L) = Le_\infty + \int_{-q}^{q} 2\lambda \rho_\infty g_2(\lambda)\, \mathrm{d}\lambda - \frac{q}{6L\rho_\infty(q)}. \tag{24}$$

To proceed further we define the dressed particle energy $\varepsilon(\lambda)$ [1] which satisfies

$$\begin{aligned} &[(1 - \hat{K}/2\pi)\varepsilon](\lambda) = \lambda^2 - \zeta \\ &\varepsilon(\pm q) = 0 \end{aligned} \tag{25}$$

and the velocity of sound [1]

$$V_s \equiv \frac{\partial \varepsilon(\lambda)}{\partial P(\lambda)}\bigg|_{\lambda=q} = \frac{2q + \int_{-q}^{q} 2\lambda M(\lambda, q)\,\mathrm{d}\lambda}{2\pi\rho_\infty(q)} \tag{26}$$

where $P(\lambda)$ is the dressed particle momentum. We get finally

$$E_0(L) = Le_\infty - \frac{\pi V_s}{6L} + \mathrm{O}\left(\frac{1}{L^2}\right). \tag{27}$$

Since we know from the correlation functions [15] that $N_f = V_s$ we can immediately identify $c = 1$.

Now we proceed to find the finite-size corrections to the excited-state energies and momenta.

We are mainly interested in states which become gapless in both energy and momentum in the thermodynamic limit. Such states can be produced in two ways. Firstly we can remove particles from the 'Fermi' sea and place them outside to create particle-hole pairs near $+q$ and $-q$. Secondly we can add some particles to the ground state.

We can also consider states which are gapless in energy but not in momentum by shifting all the integers (or half-odd integers) by an integer. Such states produce oscillatory behaviour superimposed on power law decays in the correlation functions.

The most general excited state is produced as follows. We add r particles to obtain the ground state for $(N+r)$ particles. We then shift all the integers characterising the roots by t. Finally, we create particle-hole pairs $(n_p n_h)$ near $\pm q$ (labelled by $s_\pm$) at the following positions:

$$\begin{aligned} n_p(s_\pm) &= \pm\tfrac{1}{2}(N-1+r) + t \pm n_\pm(s_\pm) \\ n_h(s_\pm) &= \pm\tfrac{1}{2}(N-1+r) + t \mp m_\pm(s_\pm). \end{aligned} \tag{28}$$

The dimension and spin of these states are obtained by methods described above to be

$$x = \left(r^2 x_p + \sum_{s+} (n_+(s_+) + m_+(s_+)) + \sum_{s-} (n_-(s_-) + m_-(s_-)) + \frac{t^2}{4x_p} \right) \tag{29}$$

$$S = 2rt + \sum_{s+} (n_+(s_+) + m_+(s_+)) - \sum_{s-} (n_-(s_-) + m_-(s_-))$$

$$x_p = \tfrac{1}{4}(1 + C(q))^2$$

$$C(\lambda) = -(1/2\pi)(1 + \hat{M})(\theta(\lambda - q) + \theta(\lambda + q))$$

where the coefficient of t^2 in x has been identified as $1/4x_p$ to order $1/g^5$.

We can now identify a few operators associated with these states. The marginal operator is associated with the state $t = r = 0$ and one particle-hole pair near $+q$ and $-q$ each with $n_+(s_+) = n_-(s_+) = 1$ and $m_\pm(s_\pm) = 0$. It has dimension 1 and spin 0. The current operator is associated with the following three lowest states: (a) $t = r = 0$, $n_\pm(s_\pm) = 1$; $m_\pm = n_\mp = 0$, with dimension 1 and spin ± 1. (b) $t = 1$, $r = n_\pm = m_\pm = 0$, with dimension $1/4x_p$, spin 0 and producing oscillatory behaviour because it has a macroscopic momentum. The field operator is associated with $r = 1$, $t = n_\pm = m_\pm = 0$ and has dimension x_p and spin zero.

Our exact result for x_p agrees with that of Popov [15] up to fifth order in a $(1/g)$ expansion. We hope to show this equivalence exactly.

The general excitation is similar to a spin-wave excitation with quantum number r combined with a vortex excitation with quantum number t in the Gaussian model [16].

We have also calculated some higher-order corrections to the energies and found only integer powers of $1/L$ as opposed to the XXZ chain [10, 14]. The physical reason for this is that only operators which conserve particle number and have no macroscopic momentum can appear in the difference between the fixed point and starting Hamiltonians. This corresponds to operators in the $r=0$, $t=0$ sector which contains only integer dimensions in the NLS model. In the XXZ model, this sector contains string excitations with non-integer dimensions and these appear in the corrections to the ground-state energy [10, 14].

We believe that this technique can be generalised to non-trivial boundary conditions in a straightforward manner. One can also get more precise information about the coefficients in the operator product expansion and the irrelevant operators in the starting Hamiltonian. This information can be used to get the corrections to scaling for the Green functions and to ascertain the region of validity of asymptotic conformal behaviour. We hope to present these results in a forthcoming publication.

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Note added. After the completion of this work it was brought to our attention that similar work appears in [17]. However, general excitations are not considered and it is unclear whether their method can be extended to get all higher-order finite-size corrections since no details are given.

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